1. Consider a game in which a fair coin is tossed repeatedly until the first head appears. Let $N$ be the number of tosses needed to get the first head.

(a) Show that $P(N = n) = \frac{1}{2^n}, n = 1, 2, 3, \ldots$

(b) Find $E[N]$ and $Var(N)$.

(c) Consider the random prospect that, if the first head appears on the $n$th toss, you will get $2^n$ dollars. How much would you be willing to pay for this random prospect?

2. When $X$ is a nonnegative random variable with distribution function $F_X(b)$, so that $E[X]$ exists, show that

$$ E[X] = \int_0^\infty (1 - F_X(b)) \, db. $$

(Use the fact that $E[X]$ exists if and only if $\lim_{x \to \infty} x[1 - F_X(x)] = 0$.

3. Show that $Var(X) = E[X^2] - (E[X])^2$.

4. Find $E[X]$ and $Var(X)$ for the following random variables:

(a) uniformly distributed on the interval $(a, b)$.

(b) exponential with mean $1/\lambda$.

5. Suppose $X \sim \text{Geometric}(p)$, so that $f_X(x) = p(1-p)^{x-1}$ for $x = 1, 2, 3, \ldots$ Show that $f_X$ is a discrete density and find $E[X]$ and $Var(X)$.

6. The median of a random variable $X$ is that value $m$ such that $F_X(m) = \frac{1}{2}$. Find the median of $X$ if $X$ is

(a) uniformly distributed on the interval $(a, b)$.

(b) exponential with mean $1/\lambda$.

7. Compute the hazard rate function of the Weibull random variable $X$. The Weibull distribution function with parameters $\nu, \alpha,$ and $\beta$ is

$$ F_X(x) = \begin{cases} 0 & \text{for } 0 < x \leq \nu \\
1 - \exp \left\{ -\left( \frac{x - \nu}{\alpha} \right)^\beta \right\} & \text{for } x > \nu. \end{cases} $$