Consider the subgroup lattice of $A_4$.

Notice that the 3-subgroups of $A_4$ are $\langle 234 \rangle$, $\langle 134 \rangle$, $\langle 124 \rangle$, and $\langle 123 \rangle$. We can arrange these in **ascending chains**, for example,

$$\{1\} \subseteq \langle 234 \rangle \subseteq A_4$$

1. What are the 2-subgroups of $A_4$?

2. Arrange the 2-subgroups of $A_4$ in ascending chains.

Let $G$ be a group with order $p^n \cdot n$, where $(p, n) = 1$ and $p$ is a prime. We say that a maximal proper $p$-subgroup is a **Sylow $p$-subgroup** of $G$. Let $n_p$ denote the number of Sylow $p$-subgroups of $G$.

3. Identify all the Sylow 2-subgroups and Sylow 3-subgroups of $A_4$.

4. Identify the Sylow 2-subgroups and Sylow 3-subgroups of the other groups of order 12.

5. For each group of order 12, could you say that there is a unique Sylow 2-subgroup? Sylow 3-subgroup?

6. For each group of order 12, what is the order of a Sylow 2-subgroup? Sylow 3-subgroup?

7. Calculate $n_p$, $p = 2, 3$ for the 5 groups of order 12. Then complete the following sentences:

Suppose $|G| = 12$.

(a) Then $G$ has either ____________ or ____________ Sylow 3-subgroups, which are all of order ____________.

(b) Then $G$ has either ____________ or ____________ Sylow 2-subgroups, which are all of order ____________.

8. You should not have any blanks filled in with the number 0 in problem 7. Do you think that, for some group of order other than 12, we could have $n_p = 0$? Justify your answer.
9. Let $G$ be a group with order $p^n$, where $(p, n) = 1$. Write a conjecture that relates $n_p$, the number of Sylow $p$-subgroups of $G$, to $|G|$.

10. Write a conjecture about the Sylow $p$-subgroups of a group, if $n_p > 1$.

11. Write a conjecture about the unique Sylow $p$-subgroup of a group when $n_p = 1$.

12. Now consider the groups of order 6: $\mathbb{Z}_6$ and $S_3$. Do these groups satisfy your conjectures?
Lattice Diagrams of Remaining Groups of Order 12