In this lab, we are going to explore the field of quotients of the integers. Consider the set

\[ S = \{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \}. \]

Unfortunately, there are many different representations for the same quotient. For example,

\[ \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots \]

We would like to relate all of the same representations of \( \frac{a}{b} \).

1. Define a relation on the set \( S \) so that \( \frac{a}{b} \sim \frac{c}{d} \) for all quotients with the same representation.

2. Prove that your relation indeed defines an equivalence relation.

3. In proving that the relation is an equivalence relation, which properties of \( \mathbb{Z} \) did you require?

4. Consider the set of equivalence classes

\[ Q = \{ [\frac{a}{b}] \mid a, b \in \mathbb{Z}, b \neq 0 \}. \]

What is the set \( Q \)?

5. With this characterization of \( Q \) in mind, define operations of addition and multiplication on the set \( Q \).