Math 3721
Subgroups and Cyclic Groups

The most elementary groups are those that can be constructed from a single element. These are called cyclic groups. We define \( \mathbb{Z}_8 \) to be the group formed by the set \( \{0, 1, 2, 3, 4, 5, 6, 7\} \) with the operation \( * \) of addition modulo 8. If we consider the element \( 2 \in \mathbb{Z}_8 \), we can compute

\[
2 \ast 2 = 4 \\
2 \ast 2 \ast 2 = 6 \\
2 \ast 2 \ast 2 \ast 2 = 8 = 0
\]

That is, 2 generates the set \( S = \{0, 2, 4, 6\} \).

1. Show explicitly that the set \( S \) is a subgroup by showing that \( S \) is closed under the operation \( * \) and that the inverse of every element in \( S \) is also in \( S \).

2. Compute other subgroups of \( \mathbb{Z}_8 \). Record your data in a chart with headings

<table>
<thead>
<tr>
<th>element ( x \in \mathbb{Z}_8 )</th>
<th>subgroup</th>
</tr>
</thead>
</table>

Notice that several elements generate the whole group \( \mathbb{Z}_8 \). When this happens, we say that the group is cyclic and call each element that gives us the whole group a generator of the group.

Let’s try this again with \( S_3 \). Recall that

\[ S_3 = \{\rho_0, \rho, \rho^2, \mu_1, \mu_2, \mu_3\} \]

As before, we calculate the powers of an element to generate a cyclic subgroup. For example, if we choose \( \rho \in S_3 \), we have

\[
\rho^1 = \rho \\
\rho^2 = \rho \ast \rho = \rho^2 \\
\rho^3 = \rho \ast \rho \ast \rho = \rho_0
\]

Thus \( \langle \rho \rangle = \{\rho_0, \rho, \rho^2\} \). The notation \( \langle a \rangle \) stands for the cyclic subgroup generated by \( a \). Thus

\[
\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}
\]

3. Generate all the subgroups that you can for \( S_3 \) by repeating this procedure for all the other elements of \( S_3 \). Record your answers in a chart as above.

4. Are there any elements which give you \( S_3 \)?

If \( G \) is a group and \( x \in G \), then the order of \( x \) is the smallest positive integer \( n \) such that \( x^n = e \), or infinite if there is no such \( n \).
5. Find the order of each element in $S_3$.

6. Find the order of each element in $\mathbb{Z}_8$.

7. Repeat the procedure for the groups $\mathbb{Z}_5$ and $\mathbb{Z}_4$. Add a column to your tables indicating the order of each element.

8. Do you notice a relationship between the order of an element $x$ and the size of the cyclic group $\langle x \rangle$?

9. Will $\mathbb{Z}_n$ always be a cyclic group?

10. Make a conjecture about the possible generators of $\mathbb{Z}_n$. 