On normal subgroups of factorizable groups

Neil Flowers
Department of Mathematics and Statistics
Youngstown State University
Youngstown, Ohio 44555
flowers@math.ysu.edu

Thomas P. Wakefield
Department of Mathematics and Statistics
Youngstown State University
Youngstown, Ohio 44555
tpwakefield@ysu.edu

March 22, 2011

Abstract
In [5], W.R. Scott posed the question that if a nontrivial group $G$ is the product of two nilpotent subgroups, does there exist a nontrivial normal subgroup of $G$ contained in one of the nilpotent factors? J.D. Gillam showed that this does not hold in general. In this paper, we impose some conditions on the group $G$ and its factors and prove that the result will hold under these conditions. Let $F(G)$ denote the Fitting subgroup of $G$. We show that if $G = HK$ with $H$ and $K$ nilpotent, $H$ a maximal subgroup of $G$, then there is a nontrivial normal subgroup of $G$ contained in $H$ or $K$. We also establish the result when $G = HK$, $H$ and $K$ nilpotent, $F(G) = O_p(G)$ is abelian and there exists a maximal subgroup $M$ of $G$ such that $O_p(G) \cap M = 1$.

AMS Classification: Primary 20D25; Secondary 20D10
Keywords: factorizable groups, nilpotent factors

1 Introduction

In [5], W.R. Scott shows several results concerning factorizable groups and normal subgroups. In particular, in Theorem 13.3.3, it is shown that if the nontrivial finite group $G = AB$, where $A$ and $B$ are abelian subgroups, then there exists $H \leq G$ such that $1 \neq H$ and $H \leq A$ or $H \leq B$. Thus a nontrivial normal
subgroup of $G$ lies in a factor of $G$. At the end of the proof, W.R. Scott comments that it is unclear whether the result holds in the case where $A$ and $B$ are infinite abelian groups, finite nilpotent subgroups, or even finite $p$-groups. As shown in [1], the result is negative in general when $G$ is the product of finite $p$-groups. In particular, in Theorem 2 of [1], a $p$-group $P = AB$ of order $p^6$ with $\text{Core}(A) = \text{Core}(B) = 1$ is constructed, which shows that it is not possible for a proper normal subgroup of $P$ to be contained in $A$ or $B$. There are positive results for the conjecture if $G$ is the finite product of two abelian groups [4] and for a product of two abelian groups, one of which is finitely generated [6] or has finite abelian sectional rank [8]. However, in [7], Vasil'ev shows that the result does not hold for the product of two abelian subgroups in general. Establishing Theorem 1 of [3], O.V. Gorodnik explores this topic further and finds that if $G$ is solvable and has finite abelian section rank and is factored as $G = AB$ with $A$ abelian and $B$ nilpotent, then the result holds. In Theorem 3, he shows that the result holds if the solvable group $G$ is a product of a cyclic subgroup $A$ and nilpotent subgroup $B$. It is our goal to establish some results concerning the existence of such a normal subgroup in the case where $A$ and $B$ are finite nilpotent subgroups.

In this paper, all groups are finite. Let $F(G)$ denote the Fitting subgroup of $G$. We establish Theorem 1.1 and Theorem 1.2 in this paper. Theorem 1.1 shows that if $G = HK$ with $H$ and $K$ nilpotent, and $H$ is a maximal subgroup of $G$, then there is a nontrivial normal subgroup of $G$ contained in $H$ or $K$. In Theorem 1.2, we establish the result when $G = HK$, $H$ and $K$ are nilpotent, $F(G) = O_p(G)$ is abelian and there exists a maximal subgroup $M$ of $G$ such that $O_p(G) \cap M = 1$. In particular, we will prove the following.

**Theorem 1.1.** Suppose $G = HK$ where $H$, $K$ are nilpotent and $H$ is a maximal subgroup of $G$. Then there exists a nontrivial normal subgroup $1 \neq N \trianglelefteq G$ such that $N \leq H$ or $N \leq K$.

**Theorem 1.2.** Suppose $G = HK$ where $H$, $K$ are nilpotent, $F(G) = O_p(G)$ is abelian and there exists a maximal subgroup $M$ of $G$ such that $O_p(G) \cap M = 1$. Then there exists a nontrivial normal subgroup $1 \neq N \trianglelefteq G$ such that $N \leq H$ or $N \leq K$.

### 2 Preliminary Results

Our proofs of Theorems 1.1 and 1.2 rely upon the following lemmas. The first was established by Wielandt and generalized by Kegel. It appears as Theorem 13.2.9 in [5].

**Lemma 2.1.** If a finite group is the product of two nilpotent subgroups, then it is solvable.

The following lemmas present results on coprime actions. The first is Theorem 5.2.3 in [2] while the second is Theorem 5.3.15 in [2].
Lemma 2.2. Let $A$ be a $p'$-group of automorphisms of the abelian group $P$. Then

$$P = C_P(A) \times [P, A].$$

Lemma 2.3. Let $A$ be a $p'$-group of automorphisms of a $p$-group $P$ and let $N$ be an $A$-invariant normal subgroup of $P$. Then

$$C_{P/N}(A) = \frac{C_P(A)N}{N}.$$ 

3 Proofs of Main Results

We now prove Theorem 1.1.

Theorem 1.1. Suppose $G = HK$ where $H, K$ are nilpotent and $H$ is a maximal subgroup of $G$. Then there exists a nontrivial normal subgroup $1 \neq N \unlhd G$ such that $N \leq H$ or $N \leq K$.

Proof. We will proceed by contradiction. As $\cap_{g \in G} H^g \leq G$ and $\cap_{g \in G} H^g \leq H$, $\cap_{g \in G} H^g = 1$ and, similarly, $\cap_{g \in G} K^g = 1$.

As $G = HK$ and $H$ and $K$ are nilpotent, $G$ must be solvable by Lemma 2.1. We claim there exists a prime $p$ such that $O_p(G) \neq 1$. Let $1 \neq N \leq G$ be a minimal normal subgroup of $G$. If $G = N$, then $G$ is simple and, as $G$ is solvable, we must have that $G$ is abelian. Hence $G \cong \mathbb{Z}_p$ so $H = 1$ and $G = K$, a contradiction. Hence $N \neq G$. As $N \neq G$ and $G$ is solvable, $N$ is an elementary $p$-group. Thus $1 \neq N \leq O_p(G)$.

If $Q \in \text{Syl}_p(H)$, as $H$ is nilpotent, $Q \unlhd H$. Hence $H \leq N_G(Q) < G$ and so $H = N_G(Q)$ by the maximality of $H$. Thus $Q \in \text{Syl}_p(N_G(Q))$, which implies $Q \in \text{Syl}_p(G)$. Hence $H \in \text{Hall}_p(G)$. If $p \mid |H|$, let $H_p \in \text{Syl}_p(H)$. Then $H_p \in \text{Syl}_p(G)$ so that $O_p(G) \leq H_p \leq H$ and $O_p(G) \unlhd G$, a contradiction.

Hence $p \nmid |H|$. But then $|G|_p = |K|_p$. Let $K_p \in \text{Syl}_p(K)$. Then we get $O_p(G) \leq K_p \leq K$ and $O_p(G) \unlhd G$, a contradiction.

We will now establish Theorem 1.2.

Theorem 1.2. Suppose $G = HK$ where $H, K$ are nilpotent, $F(G) = O_p(G)$ is abelian and there exists a maximal subgroup $M$ of $G$ such that $O_p(G) \cap M = 1$. Then there exists a nontrivial normal subgroup $1 \neq N \unlhd G$ such that $N \leq H$ or $N \leq K$.

Proof. First we will show that $G = O_p(G)M$. Since $O_p(G) \neq 1$ and $O_p(G) \cap M = 1$, $O_p(G) \nleq M$. Hence $M < MO_p(G) \leq G$, so that $G = O_p(G)M$ by the maximality of $M$. Since $G = HK$ and $H$ and $K$ are nilpotent, Lemma 2.1 implies $G$ is solvable. Hence $C_G(O_p(G)) = C_G(F(G)) \leq F(G) = O_p(G)$. Thus $C_G(O_p(G)) \leq O_p(G)$. 

3
Let $P \in \text{Syl}_p(H)$, $R \in \text{Hall}_p(H)$, $Q \in \text{Syl}_p(K)$, and $S \in \text{Hall}_p(K)$. Since $G = HK$ and $H$ and $K$ are nilpotent, $PQ \in \text{Syl}_p(G)$ and $RS \in \text{Hall}_p(G)$. Hence

$$M \cong M/1 = M/(M \cap O_p(G)) \cong MO_p(G)/O_p(G) = G/O_p(G).$$

Thus $|G|_p = |M|_p$. Since $G$ is solvable, assume, without loss of generality, that $RS \leq M$. In addition $O_p(M) \cong O_p(G/O_p(G)) = 1$. Since $H$ and $K$ are nilpotent, $[PQ, R \cap S] = 1$. But $O_p(G) \leq G$ is a $p$-group so $O_p(G) \leq PQ$ implies $[O_p(G), R \cap S] = 1$. Hence $R \cap S \leq C_G(O_p(G)) \leq O_p(G)$, and so $R \cap S = 1$ as $R \cap S$ is a $p'$-group.

Let $T = O_{p'}(M)$. Now $F(M)$ is a $p'$-group as $O_p(M) = 1$. Hence $F(M) \leq O_{p'}(M) = T$. Since $M$ is solvable, $C_M(T) \leq C_M(F(M)) \leq F(M) \leq T$. Thus $C_M(T) \leq Z(T)$. As $Z(T) \leq C_M(T)$, we have $C_M(T) = Z(T)$.

If $C_G(T) \neq M$, then $M < MC_G(T) \leq G$ so that $G = MC_G(T)$ by the maximality of $M$. Then $T \leq MC_G(T) = G$. But $O_p(G) \leq G$ implies $[T, O_p(G)] \leq T \cap O_p(G) = 1$. Then $T \leq C_G(O_p(G)) \leq O_p(G)$. Hence $T = 1$ so that $O_{p'}(M) = 1$. But then $F(M) \leq O_p(M)O_{p'}(M) = 1$ and $M$ is solvable implies $M \cong Z_q$. Since $M$ is maximal in $G$, $|G|_q = |M|$ and $M \in \text{Syl}_q(G)$. Since $G = HK$, $q \mid |H|$ or $q \mid |K|$. Without loss of generality, assume $q \mid |H|$. Then there exists $g \in G$ such that $M \leq H^g$. By the maximality of $M$, $H^g = M$ or $H^g = G$. If $H^g = G$, then $H = G$, a contradiction as $H$ and $K$ are proper subgroups. Hence $H^g = M$ and so $H \cong Z_q$. Then $P = 1$ and so $Q \in \text{Syl}_p(G)$. Thus $O_p(G) \leq Q \leq K$ and $O_p(G) \leq G$.

Without loss of generality, $C_G(T) \leq M$. Then $C_G(T) \leq C_M(T) = Z(T) \leq C_G(T)$, so that $C_G(T) = Z(T)$.

We next assert that $H \cap K = 1$ and $|G| = |H||K|$. Since $H$ and $K$ are nilpotent, $[P \cap Q, RS] = 1$. But $T = O_{p'}(M) \leq M$ is a $p'$-group. Hence $T \leq RS$ and $[P \cap Q, T] = 1$. But $P \cap Q \leq C_G(T) = Z(T)$ and so $P \cap Q = 1$. Since $H$ and $K$ are nilpotent, $H \cap K \leq R \cap S = 1$ so that $|G| = |HK| = |H||K|$. Let $H_1 = HO_p(G) \cap M$ and $K_1 = KO_p(G) \cap M$. Then

$$HO_p(G) = HO_p(G) \cap G = HO_p(G) \cap MO_p(G) = (HO_p(G) \cap M)O_p(G) = H_1O_p(G)$$

and, similarly, $KO_p(G) = K_1O_p(G)$. Also, $H_1 \cap O_p(G) \leq M \cap O_p(G) = 1$ and $K_1 \cap O_p(G) = 1$. Now

$$H_1 \cong H_1/1 = H_1/(H_1 \cap O_p(G)) \cong H_1O_p(G)/O_p(G) = HO_p(G)/O_p(G)$$

is nilpotent and, likewise, $K_1$ is nilpotent. Also

$$[H_1]_{p'} = \left( \frac{HO_p(G)}{O_p(G)} \right)_{p'} = \left( \frac{|H|_{p'}/|O_p(G)|_{p'}}{|H \cap O_p(G)|_{p'}} \right) = |H|_{p'}.$$
Hence, since $R \leq HO_p(G) \cap M = H_1$ we get $R \in \text{Hall}_p(H_1)$ and similarly $S \in \text{Hall}_p(K_1)$. Let $L \leq H_1 \cap K_1$ be a $p$-group. Then $[L, T] \leq [L, R]S = 1$ so that $L \leq C_G(T) = Z(T)$. Hence $L = 1$ as $Z(T)$ is a $p'$-group and so $H_1 \cap K_1$ is a $p'$-group. Since $H_1$ and $K_1$ are nilpotent, $H_1 \cap K_1 \leq R \cap S = 1$, so that $H_1 \cap K_1 = 1$. Now

$$|M| \geq |H_1 K_1| = |H_1||K_1| = |HO_p(G)||KO_p(G)| |O_p(G)||O_p(G)| = |H||K| \geq |G| \geq |O_p(G)||K \cap O_p(G)| = |O_p(G)| = |M|.$$ 

Hence $M = H_1 K_1$ and $O_p(G) = (H \cap O_p(G))(K \cap O_p(G))$. Next we assert that $O_p(G) = C_{O_p(G)}(R)C_{O_p(G)}(S)$. Now

$$C_{O_p(G)}(R) = C_G(R) \cap O_p(G) = C_G(R) \cap ((H \cap O_p(G))(K \cap O_p(G))) = (H \cap O_p(G))(C_G(R) \cap K \cap O_p(G)) = (H \cap O_p(G))(C_{O_p(G)}(R) \cap K)$$

as $H$ is nilpotent. But

$$C_{O_p(G)}(R) \cap K \leq C_G(R) \cap C_G(S) \quad \text{as } K \text{ is nilpotent}$$

$$= C_G(RS) \leq C_G(T) = Z(T).$$

Hence $C_{O_p(G)}(R) \cap K = 1$ as $Z(T)$ is a $p'$-group and we get $C_{O_p(G)}(R) = H \cap O_p(G)$ and similarly $C_{O_p(G)}(S) = K \cap O_p(G)$. Thus $O_p(G) = C_{O_p(G)}(R)C_{O_p(G)}(S)$.

Consider the two cases: $[R, S] = 1$ and $[R, S] \neq 1$.


In this case, $R \leq N_G(C_{O_p(G)}(S))$ and $S \leq N_G(C_{O_p(G)}(R))$. Also $C_{O_p(G)}(R) \cap C_{O_p(G)}(S) \leq C_{O_p(G)}(RS) \leq C_G(T) = Z(T)$. Hence $C_{O_p(G)}(R) \cap C_{O_p(G)}(S) = 1$ as $Z(T)$ is a $p'$-group. By the coprime action of the $p'$-group $S$ on the $p$-group $C_{O_p(G)}(R)$, Lemma 2.2 asserts

$$C_{O_p(G)}(R) = C_{C_{O_p(G)}(R)}(S)[C_{O_p(G)}(R), S].$$
But $C_{C_{O_p(G)}(R)}(S) \leq C_{O_p(G)}(RS) \leq C_G(T) = Z(T)$, a $p'$-group. Hence $C_{C_{O_p(G)}(R)}(S) = 1$. Thus $C_{O_p(G)}(R) = [C_{O_p(G)}(R), S] \leq [O_p(G), S]$ and, by symmetry, $C_{O_p(G)}(S) = [C_{O_p(G)}(S), R] \leq [O_p(G), R]$. By the coprime action of the $p'$-groups $R$ and $S$ on $O_p(G)$, we have $O_p(G) = C_{O_p(G)}(R) \times [O_p(G), R] = C_{O_p(G)}(S) \times [O_p(G), S]$ by Lemma 2.2. Therefore $C_{O_p(G), R}(R) = 1$ and $C_{O_p(G), S}(S) = 1$. Now

$$|O_p(G)| = |O_p(G)|$$

$$|C_{O_p(G)}(R)C_{O_p(G)}(S)| = |C_{O_p(G)}(R)|O_p(G), R||$$

$$|C_{O_p(G)}(R)||C_{O_p(G)}(S)| = |C_{O_p(G)}(R)||O_p(G), R||$$

so that $|C_{O_p(G)}(S)| = ||O_p(G), R||$. Hence $C_{O_p(G)}(S) = [O_p(G), R]$ and, by symmetry, $C_{O_p(G)}(R) = [O_p(G), S]$.

Since $H$ and $K$ are nilpotent, $P \leq C_G(R)$ and $Q \leq C_G(S)$. Hence $P \leq N_G([O_p(G), R]) = N_G(C_{O_p(G)}(S))$ and $Q \leq N_G(C_{O_p(G)}(R))$. Thus $PQ \leq N_G(C_{O_p(G)}(R)) \cap N_G(C_{O_p(G)}(S))$. Now $H \cap O_p(G) = C_{O_p(G)}(R) \leq PQRS = G$ and $K \cap O_p(G) = C_{O_p(G)}(S) \leq PQSR = G$. Suppose $H \cap O_p(G) = K \cap O_p(G) = 1$. Then $O_p(G) = (H \cap O_p(G))(K \cap O_p(G)) = 1$, a contradiction to the solvability of $G$ and $G \not\cong \mathbb{Z}_p$. Hence $H \cap O_p(G) \neq 1$ or $K \cap O_p(G) \neq 1$ and we are done.

Case 2: $[R, S] \neq 1$.

In this case, $[R, S] \leq RS$ and so $[R, S]$ is a $p'$-group. If $[O_p(G), [R, S]] = 1$, then $[R, S] \leq C_G(O_p(G)) \leq O_p(G)$ so $[R, S] = 1$, a contradiction. Hence $[R, S]$ acts nontrivially on the $p$-group $O_p(G)$. Now $RS$ acts on $O_p(G)$. Let

$$O_p(G) \triangleright L_1 \triangleright L_2 \triangleright \cdots \triangleright L_n = 1$$

be a composition series of $RS$ on $O_p(G)$ such that each $L_i$ is $RS$-invariant and $RS$ acts irreducibly on $L_i/L_{i+1}$ for all $i$. If $[R, S]$ acts trivially on $L_i/L_{i+1}$ for all $i$, then the $p'$-group $[R, S]$ stabilizes a chain in the $p$-group $O_p(G)$ so that $[O_p(G), [R, S]] = 1$, a contradiction. Hence there exists $i$ such that $[R, S]$ acts nontrivially on $\overline{N} = L_i/L_{i+1}$.

As $\Phi(\overline{N}) \operatorname{char} \subset \overline{N}$, $\Phi(\overline{N})$ is $RS$-invariant. Hence $\Phi(\overline{N}) = 1$ and, since $\overline{N}$ is a $p$-group, $\overline{N}$ is an elementary $p$-group. Now $N \leq O_p(G) = C_{O_p(G)}(R)C_{O_p(G)}(S)$ and so

$$\overline{N} \leq O_p(G) = C_{O_p(G)}(R)C_{O_p(G)}(S)$$

$$= C_{O_p(G)}(R)C_{O_p(G)}(S)$$

by Lemma 2.3 through the coprime action of the $p'$-groups $R$ and $S$ on $O_p(G)$. 

6
Let $\pi = \pi \in \bar{\mathcal{N}}$. Then $\nabla \geq [\pi, R] = [\pi \gamma, R] = [\gamma, R]$. Hence

$$\bar{\gamma} \mathcal{N} \in C_{O_p(G)}/\pi(RS) \leq \frac{C_{O_p(G)(RS)} \mathcal{N}}{\mathcal{N}} \leq \frac{C_{O_p(G)}(RS) \mathcal{N}}{\mathcal{N}} \leq \frac{C_{O_p(G)}(RS) \mathcal{N}}{\mathcal{N}} \mathcal{N} = \mathcal{N}.$$

Hence $\bar{\gamma} \in C_{\mathcal{N}}(S)$, which implies $\pi \in C_{\mathcal{N}}(R)$. Thus $\nabla = C_{\mathcal{N}}(R)C_{\mathcal{N}}(S)$.

If $C_{\mathcal{N}}(S) = 1$, then $\nabla = C_{\mathcal{N}}(R)$ which implies $[\nabla, R] = 1$. Thus $[\nabla, R, S] = 1$. Also, since $\nabla$ is $S$-invariant, $[S, \nabla, R] = 1$. Hence, by the Three Subgroups Lemma, we get $[R, S, \nabla] = 1$, a contradiction. Hence $C_{\mathcal{N}}(S) \neq 1$ and similarly $C_{\mathcal{N}}(R) \neq 1$.

If $\nabla = [\nabla, R]$ by Lemma 2.2 through the coprime action of the $p'$-group $R$ on the abelian $p$-group $\nabla$, we have

$$\nabla = C_{\mathcal{N}}(R) \times [\nabla, R].$$

Hence $C_{\mathcal{N}}(R) = C_{\mathcal{N}}(R) \cap \nabla = C_{\mathcal{N}}(R) \cap [\nabla, R] = 1$ which implies $C_{\mathcal{N}}(R) = 1$, which is a contradiction. Therefore $[\nabla, R] < \nabla$ and, similarly, $[\nabla, S] < \nabla$.

Let $[\nabla, R] \leq \nabla_0$, where $\nabla_0$ is a maximal subgroup of $\nabla$. Now $\nabla$ is a $p$-group so $\nabla_0 \leq \nabla$ and $[\nabla]/[\nabla_0] = p$. As $[\nabla_0, R] \leq [\nabla, R] \leq \nabla_0$, $R \leq N_G(\nabla_0)$. Further $\nabla_0 \cap C_{\mathcal{N}}(S)^{RS}$ is $RS$-invariant and

$$\nabla_0 \cap C_{\mathcal{N}}(S)^{RS} = \nabla_0 \cap C_{\mathcal{N}}(S)^R \leq [\nabla_0, R] \leq \nabla_0 \nabla \leq \nabla$$

implying $\nabla_0 \cap C_{\mathcal{N}}(S)^{RS} \nabla$ and is $RS$-invariant. Hence $\nabla_0 \cap C_{\mathcal{N}}(S)^{RS} = 1$ so that $\nabla_0 \cap C_{\mathcal{N}}(S) = 1$.

Now $C_{\mathcal{N}}(S) \neq 1$, $[\nabla]/[\nabla_0] = p$ and $\nabla_0 \cap C_{\mathcal{N}}(S) = 1$ implies that $|C_{\mathcal{N}}(S)| = p$ and, similarly, $|C_{\mathcal{N}}(R)| = p$. Now $|C_{\mathcal{N}}(S)| = |C_{\mathcal{N}}(R)| = p$ and $C_{\mathcal{N}}(S) \neq C_{\mathcal{N}}(R)$ implies $C_{\mathcal{N}}(S) \cap C_{\mathcal{N}}(R) = 1$. Thus $[\nabla] = p^2$.

By the coprime action of the $p'$-group $R$ on the abelian $p$-group $\nabla$, Lemma 2.2 asserts $\nabla = C_{\mathcal{N}}(R) \times [\nabla, R]$. Hence $[\nabla, R] = p$ and $R \leq N_G([\nabla, R])$. But this induces a homomorphism $\phi : R \to \text{Aut}([\nabla, R])$. As $[\nabla, R]$ is cyclic, $\text{Aut}([\nabla, R])$ is abelian. Therefore, by the First Isomorphism Theorem, $R/C_R([\nabla, R])$ is abelian. Hence $R' \leq C_R([\nabla, R])$ which implies $R' \leq C_R(\nabla)$ as $\nabla = C_{\mathcal{N}}(R) \times [\nabla, R]$.

Let $\bar{\mathcal{R}}S = RS/C_{RS}(\nabla) = \bar{\mathcal{R}}S$. Now

$$\bar{\mathcal{R}} = \frac{RC_{RS}(\nabla)}{C_{RS}(\nabla)} \simeq \frac{R}{R \cap C_{RS}(\nabla)} = \frac{R}{C_R(\nabla)},$$

7
which is abelian. Similarly, \( \tilde{S} \) is abelian. By Theorem 13.3.3 in [5], there exists \( 1 \neq \tilde{L} \leq \tilde{R} \tilde{S} \) such that, without loss of generality, \( \tilde{L} \leq \tilde{R} \). Then \( \tilde{L} \leq \tilde{R} \tilde{S} \) and so \( C_{\tilde{N}}(L) \leq \tilde{N} \) is \( RS \)-invariant. Thus \( C_{\tilde{N}}(L) = 1 \) or \( C_{\tilde{N}}(L) = \tilde{N} \). If \( C_{\tilde{N}}(L) = \tilde{N} \), then \( L \leq C_{RS}(\tilde{N}) \) which implies \( \tilde{L} = 1 \), a contradiction. If \( C_{\tilde{N}}(L) = 1 \), then \( \tilde{L} \leq \tilde{R} \) implies \( L \leq RC_{RS}(\tilde{N}) \). Hence \( 1 = C_{\tilde{N}}(L) \geq C_{\tilde{N}}(RC_{RS}(\tilde{N})) \geq C_{\tilde{N}}(R) \neq 1 \), a contradiction.

\[ \square \]

References


